Ice Sheet System model

Ice flow models

Eric Larour[1], Eric Rignot[1,3], Mathieu Morlighem[1,2], **Hélène Seroussi**[1,2] Chris Borstad[1], Feras Habbal[1,3], Daria Halkides[1,4], Behnaz Khakbaz[1], John Schiermeier[1], Nicole Schlegel[1]

[1] Jet Propulsion Laboratory - California Institute of Technology

[2] Laboratoire MSSMat, École Centrale Paris, France

[3] University of California, Irvine

[4] Joint Institute for Regional Earth System Science & Engineering, UCLA

Outline

Ice Sheet flow equations

Incompressibility

$$\forall \boldsymbol{x} \in \Omega \qquad \nabla \cdot \boldsymbol{v} = \mathrm{Tr}\left(\dot{\boldsymbol{\varepsilon}}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

- $\boldsymbol{v} = (u, v, w)$ ice velocity (m/yr)
- $\dot{\boldsymbol{\varepsilon}}$ strain rate tensor (yr^{-1})

Incompressible viscous fluid

$$\boldsymbol{\sigma}' = 2\mu\dot{\boldsymbol{\varepsilon}} \tag{2}$$

- $\boldsymbol{\sigma}'$ deviatoric stress
- μ ice viscosity
- $\dot{\boldsymbol{\varepsilon}}$ strain rate tensor

Glen's flow law

$$\mu = \frac{B}{2\,\dot{\varepsilon}_e^{\frac{n-1}{n}}} \tag{3}$$

- B ice hardness
- n Glen's law coefficient ($n = 3$)
- $\dot{\varepsilon}_e$ effective strain rate (second invariant)

Ice Sheet flow equations

Conservation of momentum

$$\forall \boldsymbol{x} \in \Omega \qquad \nabla \cdot \boldsymbol{\sigma}' - \nabla P + \rho \boldsymbol{g} = \boldsymbol{0} \tag{4}$$

Assumptions:

1. Stokes flow (quasi-static assumption)
2. Coriolis effect negligible

Boundary conditions

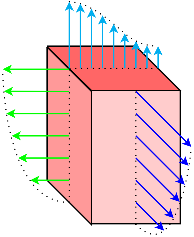
Ice/Air interface: Free surface	Γ_s	$\boldsymbol{\sigma} \cdot \boldsymbol{n} = P_{atm}\ \boldsymbol{n} \simeq \boldsymbol{0}$
Ice/Ocean interface: water pressure	Γ_w	$\boldsymbol{\sigma} \cdot \boldsymbol{n} = P_w\ \boldsymbol{n}$
Ice/Bedrock interface (1): lateral friction	Γ_b	$(\boldsymbol{\sigma} \cdot \boldsymbol{n} + \beta \boldsymbol{v})_{\parallel} = \boldsymbol{0}$
Ice/Bedrock interface (2): impenetrability	Γ_b	$\boldsymbol{v} \cdot \boldsymbol{n} = 0$
Side boundaries: Dirichlet	Γ_u	$\boldsymbol{v} = \boldsymbol{v}_{obs}$

Models description

Full-Stokes model:

- Momentum balance + incompressibility
- 3D model
- Four unknowns (v_x, v_y, v_z, p)

Full-Stokes (FS)



(v_x, v_y, v_z, P)

Model equations

$$\left\{\begin{array}{l}
\frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)-\frac{\partial p}{\partial x}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)-\frac{\partial p}{\partial y}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right)-\frac{\partial p}{\partial z}-\rho g=0
\end{array}\right.$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

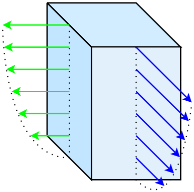
Ice flow models
Larour et al.
Ice flow equations
Approximations implemented
Ice flow equation
Diagnostic parameters
Boundary conditions
Combining models
Methods implemented in ISSM
Penalties
Tiling method
Utilization

Models description

Higher-order model:

- [Blatter, 1995, Pattyn, 2003]
- 3D model
- Horizontal and vertical velocity decoupled
- 2 (v_x, v_y) + 1 (v_z) unknowns

Blatter-Pattyn (BP)



(v_x, v_y)

Model equations

$$
\begin{cases}
\dfrac{\partial}{\partial x}\left(2\mu\dfrac{\partial v_x}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(\mu\dfrac{\partial v_x}{\partial y}+\mu\dfrac{\partial v_y}{\partial x}\right)+\dfrac{\partial}{\partial z}\left(\mu\dfrac{\partial v_x}{\partial z}+\mu\dfrac{\partial v_z}{\partial x}\right)-\dfrac{\partial p}{\partial x}=0\\
\dfrac{\partial}{\partial x}\left(\mu\dfrac{\partial v_x}{\partial y}+\mu\dfrac{\partial v_y}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(2\mu\dfrac{\partial v_y}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(\mu\dfrac{\partial v_y}{\partial z}+\mu\dfrac{\partial v_z}{\partial y}\right)-\dfrac{\partial p}{\partial y}=0\\
\dfrac{\partial}{\partial x}\left(\mu\dfrac{\partial v_x}{\partial z}+\mu\dfrac{\partial v_z}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(\mu\dfrac{\partial v_y}{\partial z}+\mu\dfrac{\partial v_z}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(2\mu\dfrac{\partial v_z}{\partial z}\right)-\dfrac{\partial p}{\partial z}-\rho g=0
\end{cases}
$$

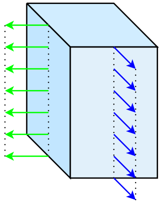
$$
\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0
$$

Models description

MacAyeal-Morland (SSA)

Shelfy-stream approximation:

- [MacAyeal, 1989]
- 2D model
- Horizontal and vertical velocity decoupled
- 2 (v_x, v_y) + 1 (v_z) unknowns



Model equations

$$
\left\{
\begin{aligned}
&\frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)-\frac{\partial p}{\partial x}=0\\
&\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)-\frac{\partial p}{\partial y}=0\\
&\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right)-\frac{\partial p}{\partial z}-\rho g=0
\end{aligned}
\right.
$$

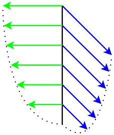
$$
\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0
$$

Models description

Hutter (SIA)

Shallow ice approximation:

- [Hutter, 1983]
- 3D analytical model
- 2 unknowns (v_x, v_y) computed separately



(v_x, v_y)

Model equations

$$\begin{cases} \frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y} + \mu\frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z} + \mu\frac{\partial v_z}{\partial x}\right) - \frac{\partial p}{\partial x} = 0 \\ \frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y} + \mu\frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z} + \mu\frac{\partial v_z}{\partial y}\right) - \frac{\partial p}{\partial y} = 0 \\ \frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z} + \mu\frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z} + \mu\frac{\partial v_z}{\partial y}\right) + \frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right) - \frac{\partial p}{\partial z} - \rho g = 0 \end{cases}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Material non-linearity

Model equations

$$\begin{cases} \frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)-\frac{\partial p}{\partial x}=0 \\ \frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)-\frac{\partial p}{\partial y}=0 \\ \frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right)-\frac{\partial p}{\partial z}-\rho g=0 \end{cases}$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Glen's flow law

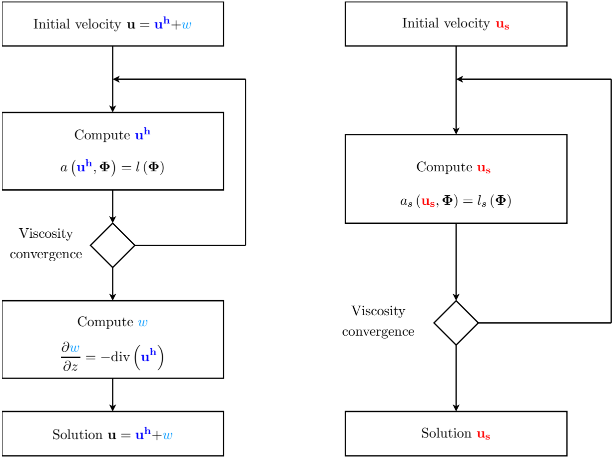
$$\mu=\frac{B}{2\,\dot{\varepsilon}_e^{\frac{n-1}{n}}} \tag{5}$$

- B ice hardness
- n Glen's law coefficient ($n=3$)
- $\dot{\varepsilon}_e$ effective strain rate (second invariant)

$\rightarrow$ Treatment of non-linearity with fixed point

Material non-linearity

Treatment of non-linearity with fixed point:



Vertical velocity computed with incompressibility for 2d shelfy-stream and 3d Blatter/Pattyn modes.

Flow equation

`setflowequation` is used to generate the approximation used to compute the velocity

- Arguments:
  1. model
  2. approximation names
  3. approximation domains
- Domains can be Argus files or array of element flags
- Approximation available
  - stokes (Full-Stokes model)
  - pattyn (Higher-order model)
  - macayeal (Shallow Shelf Approximation)
  - hutter (Shallow Ice Approximation)
- Possibility of coupling models

Flow equation

`setflowequation` is used to generate the approximation used to compute the velocity

- Examples

```
md=setflowequation(md,'hutter','all')
md=setflowequation(md,'stokes','all')
md=setflowequation(md,'macayeal','all')
md=setflowequation(md,'pattyn','all')
```

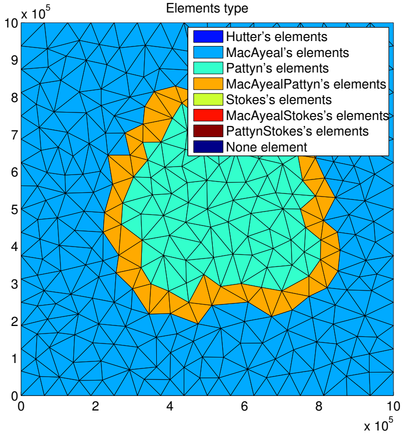
- To diplay the type of approximation:

```
>> plotmodel(md,'data','elements_type')
```

Flow equation

- To diplay the type of approximation:

```
>> plotmodel(md,'data','elements_type')
```



Flow equation class

```
>> md.flowequation

ans =

   flow equation parameters:
            ismacayealpattyn       : 0    -- is the macayeal or pattyn approximation used ?
            ishutter               : 0    -- is the shallow ice approximation used ?
            isstokes               : 0    -- are the Full-Stokes equations used ?
            vertex_equation        : N/A  -- flow equation for each vertex
            element_equation       : N/A  -- flow equation for each element
            bordermacayeal         : N/A  -- vertices on MacAyeal's border (for tiling)
            borderpattyn           : N/A  -- vertices on Pattyn's border (for tiling)
            borderstokes           : N/A  -- vertices on Stokes' border (for tiling)
```

Diagnostic class

```
>> md.diagnostic

ans =

   Diagnostic solution parameters:

     Convergence criteria:
        restol                   : 0.0001          -- mechanical equilibrium residue convergence criterion
        reltol                   : 0.01            -- velocity relative convergence criterion, NaN -> not applied
        abstol                   : 10              -- velocity absolute convergence criterion, NaN -> not applied
        maxiter                  : 100             -- maximum number of nonlinear iterations
        viscosity_overshoot      : 0               -- over-shooting constant new=new+C*(new-old)

     boundary conditions:
        spcvx                    : N/A             -- x-axis velocity constraint (NaN means no constraint)
        spcvy                    : N/A             -- y-axis velocity constraint (NaN means no constraint)
        spcvz                    : N/A             -- z-axis velocity constraint (NaN means no constraint)
        icefront                 : N/A             -- segments on ice front list (last column 0-> Air, 1-> Water, ...
                2->Ice

     Rift options:
        rift_penalty_threshold : 0                 -- threshold for instability of mechanical constraints
        rift_penalty_lock        : 10              -- number of iterations before rift penalties are locked

     Penalty options:
        penalty_factor           : 3               -- offset used by penalties: penalty = Kmax*10^offset
        vertex_pairing           : N/A             -- pairs of vertices that are penalized

     Other:
        shelf_dampening          : 0               -- use dampening for floating ice ?  Only for Stokes model
        stokesreconditioning     : 10000000000000  -- multiplier for incompressibility equation.  Only for Stokes model
        referential              : N/A             -- local referential
        requested_outputs        : N/A             -- additional outputs requested
```

Boundary conditions

Boundary conditions created automatically or manually

- Automatically:

```
>> md=SetIceSheetBC(md)
>> md=SetIceShelfBC(md,'Front.exp')
>> md=SetMarineIceSheefBC(md,'Front.exp')
```

- Manually: fields to change
  - `md.diagnostic.spcvx`
  - `md.diagnostic.spcvy`
  - `md.diagnostic.spcvz`
  - `md.diagnostic.icefront`

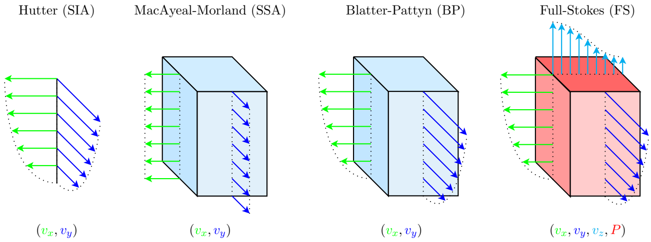
- To diplay the boundary conditions

```
>> plotmodel(md,'data','BC')
```

Models description

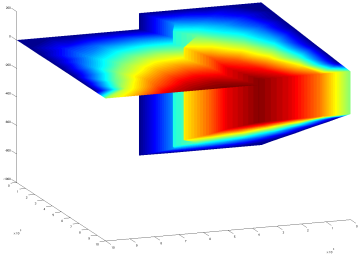
"Everything should be made as simple as possible, but no simpler." Albert Einstein

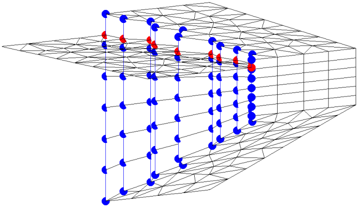
Model	Dim.	Unknowns	Reference
Full-Stokes (FS)	3d	4	[Stokes, 1845]
Blatter-Pattyn (BP)	3d	2 + 1	[Blatter, 1995, Pattyn, 2003]
Shallow shelf (SSA)	2d	2 + 1	[MacAyeal, 1989]
Shallow ice (SIA)	2d	2 + 1	[Hutter, 1983]



Penalty method

- Only to couple SSA and HO
- Very stiff spring to penalize differences between degrees of freedom



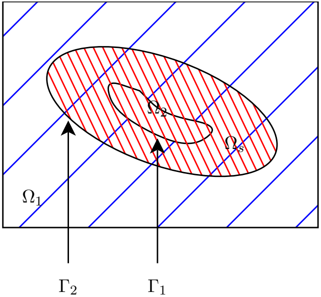


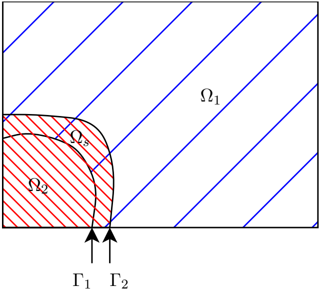
Using penalties to couple models:

```
md=setflowequation(md,'macayeal','FloatingIce.exp','fill','pattyn','coupling','penalties')
```

Domain Decomposition

- $\Omega = \Omega_1 \cup \Omega_2$
- $\Omega_S = \Omega_1 \cap \Omega_2 \neq \varnothing$
- $\boldsymbol{u} = \boldsymbol{u_1}\,|_{\Omega_1} + \boldsymbol{u_2}\,|_{\Omega_2} \in \tilde{V}(\Omega) = (V_1(\Omega_1) + V_2(\Omega_2))$



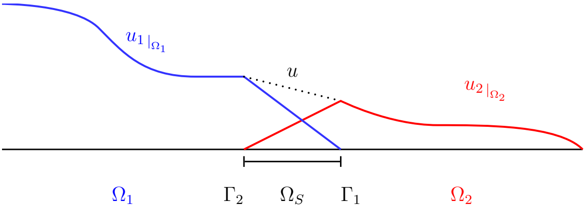


$$\text{Find } \boldsymbol{u} = \boldsymbol{u_1}\,|_{\Omega_1} + \boldsymbol{u_2}\,|_{\Omega_2} \in \tilde{V},$$

$$\forall(\boldsymbol{v_1}, \boldsymbol{v_2}) \in \tilde{V} \qquad a(\boldsymbol{u_1} + \boldsymbol{u_2}, \boldsymbol{v_1} + \boldsymbol{v_2}) = l(\boldsymbol{v_1} + \boldsymbol{v_2})$$

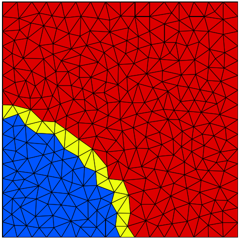
$\rightarrow$ Infinite number of solutions for the continuous problem

Discretization



We take advantage of the discretization to avoid the redundancy:

$\rightarrow$ Create one layer of elements in the superposition zone



Multi-model formulation

Two different models: a_1, a_2 and l_1, l_2

Find $\boldsymbol{u} = \boldsymbol{u_1}\,|_{\Omega_1} + \boldsymbol{u_2}\,|_{\Omega_2} \in (V_1 + V_2)$, such that:

$$\forall \boldsymbol{v} = \boldsymbol{v_1}\,|_{\Omega_1} + \boldsymbol{v_2}\,|_{\Omega_2} \in (V_1 + V_2)$$

$$\underbrace{a_1\left(\boldsymbol{u_1}\,|_{\Omega_1}, \boldsymbol{v_1}\,|_{\Omega_1}\right)}_{\text{model 1}} + \underbrace{a_2\left(\boldsymbol{u_2}\,|_{\Omega_2}, \boldsymbol{v_2}\,|_{\Omega_2}\right)}_{\text{model 2}} +$$

$$\underbrace{a_2\left(\boldsymbol{u_1}\,|_{\Omega_1}, \boldsymbol{v_2}\,|_{\Omega_2}\right) + a_1\left(\boldsymbol{u_2}\,|_{\Omega_2}, \boldsymbol{v_1}\,|_{\Omega_1}\right)}_{\text{model coupling}}$$

$$= \underbrace{l_1\left(\boldsymbol{v_1}\,|_{\Omega_1}\right)}_{\text{model 1}} + \underbrace{l_2\left(\boldsymbol{v_2}\,|_{\Omega_2}\right)}_{\text{model 2}}$$

- Coupling different mechanical models
- Easy to implement (local modification of stiffness matrices)

Flow equation

`setflowequation` is used to generate the approximation used to compute the velocity

- Examples

```
md=setflowequation(md,'pattyn',md.elementongroundedice,'fill','macayeal','coupling','penalties')
md=setflowequation(md,'pattyn',md.elementongroundedice,'fill','macayeal','coupling','tiling')
md=setflowequation(md,'stokes','Contour.exp','fill','pattyn')
```

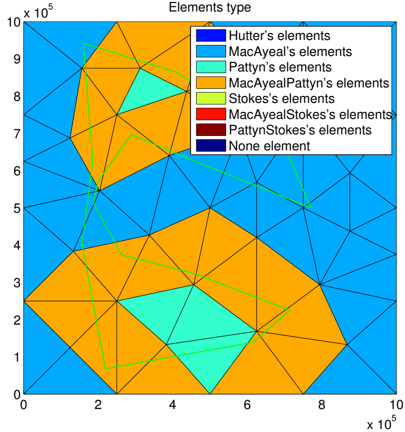
- Use `exptool` to create EXP contours

```
>> exptool('Contour.exp')
```

Flow equation

- To diplay the type of approximation:

```
>> ...
    plotmodel(md,'data','elements_type','edgecolor','k','expdisp','Contour.exp
```



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Thanks!